Int. J. Agricult. Stat. Sci. Vol. 17, Supplement 1, pp. 2005-2018, 2021 www.connectjournals.com/ijass
DocID: https://connectjournals.com/03899.2021.17.2005 ISSN : 0973-1903, e-ISSN : 0976-3392

ORIGINAL ARTICLE

STATISTICAL PROPERTIES AND APPLICATIONS OF THE REFLECTED KUMARASWAMY WEIBULL-X {EXPONENTIAL} FAMILY OF DISTRIBUTIONS

Jamal N. Al Abbasi[1], Mundher A. Khaleel[2,*] and Pelumi E. Oguntunde[3]

[1]Department of Statistics, Al Nahrain University, Iraq.
[2]Department of Mathematics, Faculty of Computer and Mathematical Sciences, Tikrit University, Iraq.
[3]Department of Mathematics, Covenant University, Nigeria.
E-mail: mun880088@tu.edu.iq

Abstract: Weibull distribution is one of the most widely applied statistical distributions in various fields, especially in reliability and lifetime analysis. It has monotonically increasing, monotonically decreasing, and constant hazard rate; these however, partly limits its applications. This limitation undoubtedly inspired researchers to develop generalized Weibull distribution that can exhibit unimodal or bathtub hazard rate. In this article, a new family of continuous distributions based on the Weibull random variable called the Reflected Kumaraswamy Weibull–X {Exponential} family of distributions was introduced. General statistical properties of the new family were established and discussed, general expressions for the ordinary and incomplete moments, the generating function, quantile function, and order statistics were also provided. The maximum likelihood method of estimation was used to estimate the model parameters. In addition, the asymptotic confidence intervals for the parameters were derived from the Fisher information matrix. Real-life application to three datasets were provided and comparison was made with respect to some existing families of distributions in the literature.

***Key words*:** Weibull family, Kumaraswamy family, Reliability, Maximum likelihood Estimation.

Cite this article

Jamal N. Al Abbasi, Mundher A. Khaleel and Pelumi E. Oguntunde (2021). Statistical properties and applications of the Reflected Kumaraswamy Weibull-X {Exponential} family of distributions. *International Journal of Agricultural and Statistical Sciences*. DocID: https://connectjournals.com/03899.2021.17.2005

1. Introduction

There has been an increased interest in defining new generators of univariate continuous distributions by adding one or more additional shape parameter(s) to the baseline distribution. In general, adding an extra shape parameter to an existing family of distribution brings about more flexibility and can be very useful in exploring tail properties. It also improves the goodness-of-fit of the generated family. In recent times, some attempts have been made to define new families of distributions that extend well-known distributions and at the same time provide great flexibility in modelling real life datasets; some of the notable examples include, and a host of others. A common feature of these generalized distributions is that, they have more parameters than the standard distributions which they generalize [Khaleel *et al*. (2020), Alizadeh *et al*. (2019), Aslam *et al*. (2018), Alizadeh *et al*. (2017), Al-Shomrani *et al*. (2016), Alizadeh *et al*. (2015), Bourguignon *et al*. (2014), Cordeiro *et al*. (2011)].

The Weibull distribution has the Exponential and Rayleigh distributions as special cases. It is one of the most commonly used distributions for modeling lifetime data and for modeling phenomenon with monotone failure rates. When modeling events with monotone hazard rates, Weibull distribution may be an initial choice because of its positively and negatively skewed density shapes. However, this distribution does not provide a good fit to data sets with bathtub shaped or upside-down bathtub shaped (unimodal) failure rates. This gap has motivated researchers to generalize the Weibull distribution, so it can exhibit unimodal or bathtub hazard

***Author for correspondence** **Received** March 19, 2021 **Revised** August 05, 2021 **Accepted** August 30, 2021

rate [Keller and Kamath (1982), Madhulika *et al*. (2020)]. Without any doubt, there is a genuine desire to search for some generalizations or modifications of the Weibull distribution that can provide more flexibility in lifetime modeling.

This research combines the works of Wallodi (1951), Alzaatreh *et al*. (2013) and Dinesh *et al*. (2021) with the aim of developing a more flexible continuous family of distributions. This paper is structured as follows. In Section 2, the Reflected Kumaraswamy Weibull- X {Exponential} family of distributions was developed, and four of its special cases were identified. Some mathematical and statistical properties of the family were provided in Section 3, estimation of the model parameters by the method of maximum likelihood was performed in Section 4 and application to real life datasets was illustrated in Section 5 followed by a concluding remark.

2. The Reflected Kumaraswamy Weibull-X{Exponential} Family of Distribution

The details on how the Reflected Kumaraswamy Weibull-X{Exponential} family of distribution developed are described in this section. Let, $r(t)$ be the probability density function (pdf) of a random variable $T \in [a, b]$, $-\infty \le a, b \le \infty$ and let $W[G(x)]$ be a function of the cumulative distribution function $G(x)$ of any random variable X such that $W[G(x)]$ satisfies the following conditions

(i) $W[G(x)] \in [a, b]$,

(ii) $W[G(x)]$ is differentiable and monotonically non-decreasing, and

(iii) $W[G(x)] \to a$ as $x \to \infty$ and $W[G(x)] \to b$ as $x \to \infty$, (1)

and If the interval $[a, b]$ is half-open or open, $W[G(x)]$ is replaced such that $\lim_{G(x) \to 0} + W(G(x)) \to a$ and $\lim_{G(x) \to 0} - W(G(x)) \to b$.

Considering the work of Alzaatreh *et al*. (2013), the cumulative distribution function (cdf) and probability density function (pdf) of the transformed-transformer family, which is otherwise known as T-X family of distributions are

$$H(x) = \int_a^{W[G(x)]} r(t)dt \tag{2}$$

and

$$h(x) = \left\{\frac{d}{dx}W(G(x))\right\} r\{W(G(x))\} \tag{3}$$

respectively.

In this paper, the cdf of the Reflected Kumaraswamy Weibull-X {Exponential}family of distributions is derived from

$$F(x) = 1 - \int_a^{V[G(x)]} r(t)dt \tag{4}$$

where, $V[G(x)]$ satisfies the condition in Equation (1), except that $V[G(x)]$ is differentiable and monotonically non-increasing. If a random variable T has the Weibull distribution with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$, then

$$r(t) = \alpha\beta^{\beta-1}e^{-\alpha t^{\beta}}\ ;\ t > 0$$

and

$$V[G(x)] = -\log_e\left(1 - \left\{1 - G^k(x;\ \xi)\right\}^{\nu}\right)$$

the definition in Equation (4) leads to the Reflected Kumaraswamy Weibull-X {Exponential} family of distributions written as 'RKW-X{Exponential}' for short with the cdf

$$F(x;\ \alpha,\ \beta,\ \kappa,\ \nu,\ \xi) = \exp\left[-\alpha\left\{-\log_e\left(1 - \left\{1 - G(x;\xi)^{\kappa}\right\}^{\upsilon}\right)\right\}^{\beta}\right] \tag{5}$$

for $x \in D \subseteq \Re;\ \alpha, \beta, \kappa, \upsilon > 0$

The corresponding pdf of the RKW-X{Exponential} family of distributions can be written as

$$f(x;\ \alpha,\beta,\kappa,\upsilon,\xi) = \alpha\beta\kappa\upsilon g(x;\xi)\frac{G(x;\xi)^{\kappa-1}\left\{1 - G(x;\xi)^{\kappa}\right\}^{\nu-1}}{1 - \left\{1 - G(x;\xi)^{\kappa}\right\}^{\nu}} \times \left\{-\log_e\left(1 - \left\{1 - G(x;\xi)^{\kappa}\right\}^{\upsilon}\right)\right\}^{\beta-1} \times \exp\left[-\alpha\left\{-\log_e\left(1 - \left\{1 - G(x;\xi)^{\kappa}\right\}^{\nu}\right)\right\}^{\beta}\right] \tag{6}$$

where, α, β, κ, ν are additional shape parameters

and ξ is a vector of parameters. The additional parameters introduced by the *RKW-X* generator are sought as a manner to furnish a more flexible distribution. Also, the expression in Equation (6) will be most tractable when the cdf $G(x, \xi)$ and pdf $g(x, \xi)$ have simple analytic expressions.

Special Cases

Some families of distributions are special cases of the *RKW-X*{Exponential} family of distributions. For instance

1. When $\nu = 1$, the RKW-X{Exponential} family reduces to $F(x) = \exp[-\alpha\{-log_e(G(x)^{\kappa})\}^{\beta}]$, which is the Reflected Exponentiated Weibull-X{Exponential} family of distributions.
2. When $\nu = \kappa = 1$, the *RKW-X*{Exponential} family reduces to $F(x) = \exp[-\alpha\{-log_e(G(x))\}^{\beta}]$, which is the Reflected Weibull-X{Exponential} family of distributions.
3. When $\beta = 1$, the *RKW-X*{Exponential} family reduces to $F(x) = [(1-\{1-(G(x)^{\kappa})\}^{\nu}]^{\alpha}$, which is the Exponentiated Kumaraswamy-G family of distributions [Silva *et al*. (2019)].
4. When $\alpha = \beta = 1$, the *RKW- X*{Exponential} family reduces to $F(x) = [(1-\{1 - G(x)^{\kappa}\}^{\nu})]$, which is the Kumaraswamy-G family of distributions [Cordeiro and de Castro (2011)].
5. When $\alpha = \beta = \nu = 1$, the *RKW-X*{Exponential} family reduces to $F(x) = G(x)^{\kappa}$, which is the Exponentiated-G family of distributions [Cordeiro *et al.* (2013)].
6. If $\alpha = \beta = \nu = \kappa = 1$, the *RKW-X*{Exponential} family reduces to $F(x) = G(x)$, the baseline G distribution.

Special Models

The pdf in Equation (6) allows greater flexibility of its tails and can be widely applied in many areas of statistics. For illustration purpose, the *RKW-X* family of distribution was used to extend some well-known standard distributions in the literature. It is good to note that the pdf will be most tractable when the cdf $G(x, \xi)$ and the pdf $g(x, \xi)$ have closed forms.

The Reflected Kumaraswamy Weibull-Uniform {Exponential} Distribution

Consider the baseline distribution as a Uniform distribution with scale parameter $\theta > 0$. The density and distribution functions of the uniform distribution are given by $g(x, \theta) = \frac{1}{\theta}$, $0 < x < \theta$ and $g(x, \theta) = \frac{x}{\theta}$, respectively. Then, the Reflected Kumaraswamy Weibull-Uniform{Exponential} (*RKW-U*{Exponential}) distribution is given by

$$f(x;\alpha,\beta,\kappa,\nu,\theta) = \frac{\alpha\beta\kappa\nu}{\theta}\frac{\left(\frac{x}{\theta}\right)^{\kappa-1}\left\{1-\left(\frac{x}{\theta}\right)^{\kappa-1}\right\}^{\nu-1}}{\left[1-\left\{1-\left(\frac{x}{\theta}\right)^{\kappa-1}\right\}^{\nu}\right]}$$

$$\times\left\{-\log_e\left(1-\left\{1-\left(\frac{x}{\theta}\right)^{\kappa}\right\}^{\nu}\right)\right\}^{\beta-1}$$

$$\times\exp\left[-\alpha\left\{-\log_e\left(1-\left\{1-\left(\frac{x}{\theta}\right)^{\kappa}\right\}^{\nu}\right)\right\}^{\beta}\right] \qquad (7)$$

and the corresponding cdf is

$$F(x;\alpha,\beta,\kappa,\nu,\xi) = \exp\left[-\alpha\left\{-\log_e\left(1-\left\{1-\left(\frac{x}{\theta}\right)^{\kappa}\right\}^{\nu}\right)\right\}^{\beta}\right] \qquad (8)$$

for $x \in D \subseteq \Re;\ \alpha,\beta,\kappa,\upsilon > 0$, where θ and α are scale parameters. β, ν, κ are shape parameters.

The Reflected Kumaraswamy Weibull-Logistic {Exponential} Distribution

Consider the parent distribution as a Logistic distribution with scale parameter $\theta > 0$, the density and distribution functions of the logistic distribution are given by $g(x, \theta) = \theta e^{-\theta x}(1+ e^{-\theta x})^{-2}$ and $g(x, \theta) = (1+ e^{-\theta x})^{-1}$ respectively. Then, the Reflected Kumaraswamy Weibull Logistic {Exponential} (RKW-Lo {Exponential}) distribution is given by

$$f(x;\alpha,\beta,\kappa,\nu,\xi) = \frac{\alpha\beta\kappa\nu}{\theta\left(1+e^{-\theta x}\right)^2}$$

$$\times e^{-\theta x}\frac{\left(1+e^{-\theta x}\right)^{-(\kappa+1)}\left\{1-\left(1+e^{-\theta x}\right)^{-\kappa}\right\}^{\nu-1}}{\left[1-\left\{1-\left(1+e^{-\theta x}\right)^{-\kappa}\right\}^{\nu}\right]}$$

$$\times\left\{-\log_e\left(1-\left\{1-\left(1+e^{-\theta x}\right)^{-\kappa}\right\}^{\nu}\right)\right\}^{\beta-1}$$

$$\times\exp\left[-\alpha\left\{\log_e\left(1-\left\{1-\left(1+e^{-\theta x}\right)^{-\kappa}\right\}^{\nu}\right)\right\}^{\beta}\right] \quad (9)$$

and the corresponding cdf is

$$F(x,\alpha,\beta,\kappa,\nu,\xi)$$

$$=\exp\left[-\alpha\left\{-\log_e\left(1-\left\{1-\left(1+e^{-\theta x}\right)^{\kappa}\right\}^{\nu}\right)\right\}^{\beta}\right] \quad (10)$$

where, θ and α are scale parameters. β, ν, κ are shape parameters.

The Reflected Kumaraswamy Weibull-Lomax {Exponential} Distribution

Consider the baseline distribution as Lomax distribution with scale parameter θ > 0 and shape parameter δ > 0. The density and distribution functions of the Lomax distribution are given by $g(x;\theta,\delta)=\frac{\delta}{\theta}\left(1+\frac{x}{\theta}\right)^{-(\delta+1)}$ and $G(x;\theta,\delta)=1-\left(1+\frac{x}{\theta}\right)^{-\delta}$, respectively. Then, the Reflected Kumaraswamy Weibull-Lomax {Exponential} (RKW-Lom {Exponential}) distribution is given by

$$f(x;\alpha,\beta,\kappa,\nu,\theta,\delta)$$

$$=\frac{\alpha\beta\kappa\nu\delta\left(1-\left(1+\frac{x}{\theta}\right)^{-\delta}\right)^{\kappa-1}\left\{1-\left(1-\left(1+\frac{x}{\theta}\right)^{-\delta}\right)^{\kappa}\right\}^{\nu-1}}{\theta\left(1+\frac{x}{\theta}\right)^{(\delta+1)}\left[1-\left\{1-\left(1-\left(1+\frac{x}{\theta}\right)^{-\delta}\right)^{\kappa}\right\}^{\nu}\right]}$$

$$\times\left\{-\log_e\left(1-\left\{1-\left(1-\left(1+\frac{x}{\theta}\right)^{-\delta}\right)^{\kappa}\right\}^{\nu}\right)\right\}^{\beta-1}$$

$$\times\exp\left[-\alpha\left\{-\log_e\left(1-\left\{1-\left(1-\left(1+\frac{x}{\theta}\right)^{-\delta}\right)^{\kappa}\right\}^{\nu}\right)\right\}^{\beta}\right] \quad (11)$$

and the corresponding cdf is

$$f(x;\alpha,\beta,\kappa,\nu,\theta,\delta)$$

$$=\exp\left[-\alpha\left\{-\log_e\left(1-\left\{1-\left(1+\frac{x}{\theta}\right)^{-\delta}\right)^{\kappa}\right\}^{\nu}\right)\right\}^{\beta}\right] \quad (12)$$

for $x\in D\subseteq\Re;\ \alpha,\beta,\kappa,\upsilon>0$, where θ and α are scale parameters. β, κ, ν, δ are shape parameters.

The Reflected Kumaraswamy Weibull-Weibull {Exponential} Distribution

As a fourth example, consider the baseline distribution as a Weibull distribution with scale parameter θ > 0 and shape parameter δ > 0. The density and distribution functions of the Weibull distribution are given by $g(x;\theta,\delta)=\theta\delta x^{\delta-1}e^{-\theta x^{\delta}}$ and $g(x)=1-e^{-\theta x^{\delta}}$, respectively. Then, the Reflected Kumaraswamy Weibull-Weibull{Exponential} (RKW-W{Exponential}) distribution density function is given by

$$f(x;\alpha,\beta,\kappa,\nu,\theta,\delta)$$

$$=\alpha\beta\kappa\nu\delta x^{\delta-1}e^{-\theta x^{\delta}}\frac{\left(1-e^{-\theta x^{\delta}}\right)^{\kappa-1}\left\{1-\left(1-e^{-\theta x^{\delta}}\right)^{\kappa}\right\}^{\nu-1}}{\left[1-\left\{1-\left(1-e^{-\theta x^{\delta}}\right)^{\kappa}\right\}^{\nu}\right]}$$

$$\times\left\{-\log_e\left(1-\left\{1-\left(1-e^{-\theta x^{\delta}}\right)^{\kappa}\right\}^{\nu}\right)\right\}^{\beta-1}$$

$$\times\exp\left[-\alpha\left\{-\log_e\left(1-\left\{1-\left\{1-\left(1-e^{-\theta x^{\delta}}\right)^{\kappa}\right\}^{\nu}\right\}^{\nu}\right)\right\}^{\beta}\right] \quad (13)$$

and the corresponding cdf is

$$f(x;\alpha,\beta,\kappa,\nu,\theta,\delta)$$

$$=\exp\left[-\alpha\left\{-\log_e\left(1-\left(1-e^{-\theta x^{\delta}}\right)^{\kappa}\right)^{\nu}\right\}^{\beta}\right] \quad (14)$$

where, θ and α are scale parameters. β, κ, ν, δ are shape parameters.

Reliability Analysis

The reliability function which is denoted by $\overline{F}(x;\xi)$

is the probability that an item does not fail before sometime x, it is defined by

$$\overline{F}(x;\xi)=1-F(x;\xi) \tag{15}$$

The reliability function of the RKW-X{Exponential} family of distribution with the parent distribution $G(x;\xi)$ is denoted by $\overline{F}_{RKW-X}(x;\alpha,\beta,\kappa,\nu,\xi)$ and can be used to characterize life time data analysis. The function is given below

$$\overline{F}_{RKW-X}(x;\alpha,\beta,\kappa,\nu,\xi) = 1-\exp\left[-\alpha\left\{-\log_e\left(1-\left(1-G(x;\xi)\right)^{\kappa}\right)^{\nu}\right\}^{\beta}\right] \tag{16}$$

for $x\in D\subseteq\Re;\ \alpha,\beta,\kappa,\upsilon>0$.

Hazard rate function is another way of characterizing a random variable. The hazard rate function can be denoted by $h(x;\xi)$ and is also known as failure rate. It is obtained using the relation

$$h_{RKW-X}(x;\xi)=\frac{f(x;\xi)}{\overline{F}(x;\xi)} \tag{17}$$

Hazard rate function of the KRW-X{Exponential} family of distribution with a parent distribution $G(x;\xi)$ is given by

$$\begin{aligned} &h_{RKW-X}(x;\alpha,\beta,\kappa,\nu,\xi) \\ &=\alpha\beta\kappa\nu g(x;\xi)\frac{G(x;\xi)^{\kappa-1}\left\{1-G(x;\xi)^{\nu-1}\right\}}{1-\left\{1-G(x;\xi)^{\kappa}\right\}^{\nu}} \\ &\quad\times\left\{-\log_e\left(1-\left(1-G(x;\xi)\right)^{\kappa}\right)^{\nu}\right\}^{\beta-1} \\ &\times\frac{\exp\left[-\alpha\left\{-\log_e\left(1-\left\{1-G(x;\xi)^{\kappa}\right\}^{\nu}\right)\right\}^{\beta}\right]}{1-\exp\left[-\alpha\left\{-\log_e\left(1-\left\{1-G(x;\xi)^{\kappa}\right\}^{\nu}\right)\right\}^{\beta}\right]} \end{aligned} \tag{18}$$

For illustration purpose, the plots for the cdf, pdf and hazard function of a special case, Reflected Kumaraswamy Weibull-Exponential Exponential (RKW-EE) distribution are presented in Figs. 1, 2 and 3, respectively.

3. General Statistical Properties of the Reflected Kumaraswamy Weibull-X{Exponential} (RKW-X{Exponential}) Family of Distributions

In this section, general statistical properties of the RKW-X{Exponential}family of distributions which include the expansion of pdf and cdf, quantile function (qf), random variable generation, moments and order statistics are derived and established.

Expansion of the pdf and cdf

By expanding the exponential function $\exp[-az^b]$ $=\sum_{m=0}^{\infty}\frac{(-1)^m a^m z^{mb}}{m!}$ in Equation (1), then

$$\exp\left[-\alpha\left\{-\log_e\left(1-\left[1-G(x;\xi)^{\kappa}\right]^{\nu}\right)\right\}^{\beta}\right]$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^m}{m!}\left\{-\log_e\left(1-G(x;\xi)^{\kappa}\right)\right\}^{m\beta}$$

$$F(x)=\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^m}{m!}\left\{-\log_e\left(1-\left[1-G(x;\xi)^{\kappa}\right]^{\nu}\right)\right\}^{m\beta}$$

And by using $\left[-\log(1-z)\right]^c=z^c+c\sum_{i=0}^{\infty}p_i(c+i)z^{c+i+1}$, for any real parameter $z\in(0,1]$;

$$F(x)=\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^m}{m!}\left[\left[1-G(x;\xi)^{\kappa}\right]^{\nu m\beta}+m\beta\sum_{i=0}^{\infty}p_i\left[m\beta+i\right]\left[1-G(x)^{\kappa}\right]^{\nu(m\beta+i+1)}\right]$$

Using general binomial expansion,

$$(1-z)^b=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(b+1)}{\Gamma(b+1-j)\,j!}z^j$$

$$F(x)=\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^m}{m!}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\nu\beta m+1)}{\Gamma(\nu\beta m+1-j)\,j!}G(x)^{kj}$$

$$+m\beta\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}p_i\left[m\beta+i\right]\frac{(-1)^j\Gamma(\nu(i+\beta m+2))}{\Gamma(\nu(i+\beta m+2-j))\,j!}G(x)^{kj}$$

Using $G(x)^{\delta}=\sum_{l=0}^{\infty}S_l(\delta)G(x)^l$,

$$F(x)=\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^m}{m!}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\nu\beta m+1)}{\Gamma(\nu\beta m+1-j)\,j!}S_i(kj)^{kj}G(x)^l$$

$$+m\beta\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}p_i[m\beta+i]\frac{(-1)^j\Gamma(\nu(i+\beta m+2))}{\Gamma(\nu(i+\beta m+2-j))\,j!}\sum_{l=0}^{\infty}S_l(kj)G(x)^l$$

where,

$$S_l(\delta)=\sum_{r=1}^{\infty}(-1)^l\binom{\delta}{r}\binom{r}{l}$$

Expanding $G(x)^{\kappa(m+1)\beta}$ and $G(x)^{\kappa[i+(m+1)\beta+1]}$ in power series, $F(x)$ can be expressed as

$$F(x)=\sum_{r=0}^{\infty}\omega_r G(x)^r=\sum_{r=0}^{\infty}\omega_r H_r(x) \quad (19)$$

where, $H_r(x)$ denotes the cdf of the Exp($-G(x)$) distribution and

$$\omega_r=\sum_{m,j=0}^{\infty}\frac{(-1)^{m+j+r+2}\alpha^{m+1}}{(m+1)!}\binom{j}{r}\binom{(m+1)\beta}{j}$$

$$+(m+1)\beta\sum_{i=0}^{\infty}p_i[(m+1)\beta+i]\binom{[i+(m+1)\beta+1]}{j}$$

The corresponding pdf can be expressed as

$$f(x)=\sum_{r=0}^{\infty}(r+1)v_r G(x)^r g(x)=\sum_{r=0}^{\infty}v_r h_{r+1}(x) \quad (20)$$

where, $h_{r+1}(x)$ denotes the pdf of the *Exp* ($-G(r+1)$) distribution and $v_r=\omega_{r+1}$. So, several properties of the RKW-X family of distribution can be obtained by knowing those of the exp ($-$ G) distribution, [Mudholkar *et al* (1995), Gupta and Kundu (2001), Nadarajah and Kotz (2006)] among others.

Moments

The obtain the moment, let $Y_{r+1}=exp\,(-G(r+1))$. A first formula for the *n*th moments of random variable *X* can be obtained from Equation (20) as

$$\mu'_n=E(X^n)=\sum_{r=0}^{\infty}v_r E(Y_{r+1}^n) \quad (21)$$

Explicit expressions for moments of several exponentiated distributions are given by (16). They can be used to produce μ'_n.

A second formula for μ'_n can be obtained in terms of the baseline quantile function $QG(u)$ as

$$\mu'_n=\sum_{r=0}^{\infty}(r+1)v_r\gamma(n,r) \quad (22)$$

where the integral can be expressed in terms of the G quantile function

$$\gamma(n,\alpha)=\int_{-\infty}^{\infty}x^n G^a(x)g(x)dx=\int_0^1 Q_G(u)^n u^a du \quad (23)$$

The ordinary moments of several RKW-X{Exponential} family of distributions can be determined directly from Equations (22) and (23).

Incomplete Moments

The incomplete moments play an important role in measuring inequality. For example, the Lorenz and Bonferroni curves, which depend upon the first incomplete moment of a distribution. The incomplete moment of *X* is calculated as

$$m_n(y)=\int_{-\infty}^{y}x^n f(x)dx=\sum_{r=0}^{\infty}(r+1)v_r\int_0^{G(y)}Q_G(u)u^r du \quad (24)$$

The last integral can be computed for most baseline G distributions.

Let, $\mu'_n=E(X^n)$ be the ordinary moment of *X* calculated from Equations (21) or (22), the descending factorial moment of *X* is

$$\mu'_{(s)}=E(X^{(t)})=E[X(X-1)\times\ldots\times(X-t+1)]$$

$$=\sum_{r=1}^{t}s(t,r)\,\mu'_{(r)}$$

where,

$$s(t,r)=\frac{1}{r!}\left[\frac{d^r y}{dx^r}x^{(t)}\right]_{x=0}$$

is the stirling number of the first kind which counts the number of ways to permute a list of t items into r cycles. So, the factorial moments can be obtained from the ordinary moments earlier given.

Generating function

Here, two formulae for the moment generating function (mgf) $M(t)=E(e^{tX})$ of *X* are provided. A first formula for $M(t)$ comes from Equation (21) as

$$M(t)=\sum_{r=0}^{\infty}v_r M_{r+1}(t) \quad (25)$$

where, $M_{r+1}(t)$ is the mgf of Y_{k+1}. Hence, $M(t)$ can be determined from the generating function of the *exp* $(-G(k+1))$ distribution.

A second formula for $M(t)$ can be derived from Equation (22) as

$$M(t)=\sum_{r=0}^{\infty}(r+1)\nu_r\rho(t,r) \tag{26}$$

$\rho(t, a)$ can be calculated from the parent quantile function $QG(x)$ as

$$\rho(t,r)=\int_{-\infty}^{\infty}e^{tx}G(x)^a g(x)dx=\int_0^1 e^{tQG(u)}u^a\,du \tag{27}$$

The moment generating functions of several special models from the RKW-X{Exponential} family of distributions can be obtained directly from Equations (26) and (27).

Quantile function, Median and Quartiles

The quantile function (QF) is denoted by $Q(u) = F^{-1}(u)$. So, the quantile function of the RKW-X family of distributions can be obtained by inverting the cdf in Equation (5) as

$$Q(u)=F^{-1}(u)=G^{-1}\left\{1-\left[1-\exp\left(-\left\{-\frac{1}{\alpha}\log_e(u)\right\}^{\frac{1}{\beta}}\right)\right]^{\frac{1}{\nu}}\right\}^{\frac{1}{\kappa}} \tag{28}$$

where, U is a uniform variate on the unit interval (0,1). The first and third quartiles of the RKW-X family of distributions can be obtained by setting $u = 0.25$ and $u = 0.75$, respectively in Equation (28), while the median can be obtained by setting $u = 0.5$.

Random variable generation

The quantile function of the RKW-X{Exponential} family of distributions has a closed form, which makes the simulation from this family easier. When the parameters α, β, κ, ν and ξ are known, random variables from the quantile function in Equation (28) can be generated as follows.

$$X=G^{-1}\left\{1-\left[1-\exp\left(-\left\{-\frac{1}{\alpha}\log_e(u)\right\}^{\frac{1}{\beta}}\right)\right]^{\frac{1}{\nu}}\right\}^{\frac{1}{\kappa}}$$

U is a uniform variate on the unit interval (0,1).

Skewness and Kurtosis

The statistical measures of skewness and kurtosis play important role in describing shape characteristics of the probability distributions. The Bowley's skewness measure based on quartiles [Kenney and Keeping (1962)] is given by

$$Skewness=\frac{Q\left(\frac{3}{4}\right)-2Q\left(\frac{1}{2}\right)+Q\left(\frac{3}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)}$$

and the Moors' kurtosis measure based on octiles [Moors (1988)] is given by

$$Kurtosis=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}$$

The previous measures skewness and kurtosis have a number of advantages compared to the classical measures of skewness and kurtosis. For instance, they are less sensitive to outliers and they exist for the distributions even without defining the moments. A more general formulation of a skewness function was described by Groeneveld and Meeden (1984) and is given by

$$Skewness=\frac{G^{-1}(u)-G^{-1}(1/2)+G^{-1}(1-u)}{G^{-1}(u)-G^{-1}(1-u)};\ \tfrac{1}{2}\le u<1$$

where, G is the cdf.

Order Statistics

If $x_{(1)}, x_{(2)}, x_{(3)}, ..., x_{(n)}$ denote the order statistics of a random sample $x_1, x_2, x_3, ..., x_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$, then the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x)=\frac{n!}{(j-1)!\,(n-j)!}f(x)[F(x)]^{j-1}[1-F(x)]^{n-j};$$

$$j = 1, 2, ..., n \tag{29}$$

Then, the pdf of the order statistics of the RKW-X{Exponential}family of distributions with random variable is given by

$$f_{X_{(j)}}(x)=\frac{n!}{(j-1)!\,(n-j)!}\alpha\beta\kappa\nu\, g(x;\xi)G(x;\xi)^{\kappa-1}$$

$$\times\left[1-G(x;\xi)^{\kappa}\right]^{\nu-1}\frac{\left\{-\log_e\left(1-\left[1-G(x;\xi)^{\kappa}\right]^{\nu}\right)\right\}^{\beta-1}}{\left\{1-\left[1-G(x;\xi)^{\kappa}\right]^{\nu}\right\}}$$

$$\times\exp\left[-\alpha\left\{-\log_e\left(1-\left[1-G(x;\xi)^{\kappa}\right]^{\nu}\right)\right\}^{\beta}\right]$$

$$\times\left\{\exp\left[-\alpha\left\{-\log_e\left(1-\left[1-G(x;\xi)^{\kappa}\right]^{\nu}\right)\right\}^{\beta}\right]\right\}^{n-j+1}$$

$$\times\left\{1-\exp\left[-\alpha\left\{-\log_e\left(1-\left[1-G(x)^{\kappa}\right]^{\nu}\right)\right\}^{\beta}\right]\right\}^{j-1} \quad (30)$$

Therefore, the pdf of the minimum order statistic $X_{(1)}$ for RKW-X{Exponential} family of distributions is given by

$$f_{X_{(1)}}(x)=n\alpha\beta\kappa\nu\, g(x;\xi)G(x;\xi)^{\kappa-1}\left[1-G(x;\xi)^{\kappa}\right]^{\nu-1}$$

$$\times\frac{\left\{\left\{-\log_e\left(1-\left[1-G(x)^{\kappa}\right]^{\nu}\right)\right\}^{\beta-1}\right\}}{\left\{1-\left[1-G(x)^{\kappa}\right]^{\nu}\right\}}$$

$$\times\left[\exp\left[-\alpha\left\{1-\left(\left\{1-G(x)^{\kappa}\right\}^{\nu}\right)\right\}^{\beta}\right]\right]^{n+1} \quad (31)$$

and the pdf of the maximum order statistic $X_{(n)}$ for RKW-X{Exponential} family of distributions is given by

$$f_{X_{(n)}}(x)=n\frac{n!}{(j-1)!\,(n-j)!}\alpha\beta\kappa\nu\, g(x;\xi)G(x;\xi)^{\kappa-1}$$

$$\times\left[1-G(x;\xi)^{\kappa}\right]^{\nu-1}\frac{\left\{-\log_e\left(1-\left[1-G(x;\xi)^{\kappa}\right]^{\nu}\right)\right\}^{\beta-1}}{\left\{1-\left[1-G(x;\xi)^{\kappa}\right]^{\nu}\right\}}$$

$$\times\left\{\exp\left[-\alpha\left\{-\log_e\left(1-\left[1-G(x)^{\kappa}\right]^{\nu}\right)\right\}^{\beta}\right]\right\}^{2}$$

$$\times-\left\{\exp\left[-\alpha\left\{-\log_e\left(1-\left[1-G(x)^{\kappa}\right]^{\nu}\right)\right\}^{\beta}\right]\right\}^{n-1} \quad (32)$$

4. Estimation of Parameter

In a view to estimating the parameters of the RKW-X{Exponential} family of distributions, the method of the maximum likelihood estimation (MLE) was used. Let, $X = (x_1, x_2, ..., x_n)^T$ be a random sample of size n from RKW – X{Exponential} $\{x; \alpha, \beta, \alpha, b, x^T\}^T$ with parameter vector $\theta = (\alpha, \beta, \alpha, b, x^T)^T$, where $\xi^T = (\xi_1, \xi_2, ..., \xi_p)^T$ corresponds to the parameter vector of the baseline distribution $G(x; \xi)$. Then the likelihood function of this sample can be written as

$$L=\prod_{i=1}^{n}\left[\alpha\beta\kappa\nu\, g(x;\xi)\frac{G(x_i;\xi)^{\kappa-1}\left\{1-G(x_i;\xi)^{\kappa}\right\}^{\nu-1}}{1-\left\{1-G(x_i;\xi)^{\kappa}\right\}^{\nu}}\right.$$

$$\times\left\{-\log_e\left\{1-G(x_i;\xi)^{\kappa}\right\}^{\nu}\right\}^{\beta-1}$$

$$\left.\times\exp\left[-\alpha\left\{-\log_e\left(1-\left\{1-G(x;\xi)^{\kappa}\right\}^{\nu}\right)\right\}^{\beta}\right]\right]$$

Hence, the log-likelihood function *logL* becomes

$$\log L=n\left[\log_e(\alpha)+\log_e(\beta)+\log_e(\kappa)+\log_e(\nu)\right]$$

$$+\sum_{i=1}^{n}\log_e\left[g(x_i;\xi)\right]+(\kappa-1)\sum_{i=1}^{n}\log_e\left[G(x_i;\xi)\right]$$

$$+(\nu-1)\sum_{i=1}^{n}\log_e\left[1-G(x_i;\xi)^{\kappa}\right]-\sum_{i=1}^{n}\log_e\left[1-\left\{1-G(x_i;\xi)^{\kappa}\right\}^{\nu}\right]$$

$$+(\beta-1)\sum_{i=1}^{n}\log_e\left[-\log_e\left[1-\left\{1-G(x_i;\xi)^{k}\right\}^{\nu}\right]\right]$$

$$-\alpha\sum_{i=1}^{n}\left\{-\log_e\left[1-\left\{1-G(x_i;\xi)^{k}\right\}^{\nu}\right]\right\}^{\beta}$$

By taking the partial derivatives of the log-likelihood function with respect to parameters α, β, κ, ν and ξ, components of the score vector $U_\theta = (U_\alpha, U_\beta, U_\kappa, U_\nu, U_{\xi T})^T$ can be obtained as follows.

$$U_{\alpha}=\frac{\partial lf}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\left[-\ln\left\{\left(1-G(x_i;\xi)^{k}\right)^{\nu}\right\}\right]^{\beta}$$

$$U_{\beta}=\frac{\partial lf}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\ln\left(-\ln\left(\left\{1-G(x_i;\xi)^{k}\right\}^{\nu}\right)\right)$$

$$-\alpha\sum_{i=1}^{n}\left[-\ln\left(\left\{1-G(x_i;\xi)^{k}\right\}^{\nu}\right)\right]^{\beta}\left[\ln\left(-\ln\left(\left\{1-G(x_i;\xi)^{k}\right\}^{\nu}\right)\right)\right]$$

$$U_{\kappa}=\frac{\partial lf}{\partial \kappa}=\frac{n}{\kappa}+\sum_{i=1}^{n}\ln\left(G\left(x_i;\xi\right)\right)+\sum_{i=1}^{n}\frac{G\left(x_i;\xi\right)^{k}\ln\left[G\left(x_i;\xi\right)\right]}{1-G\left(x_i;\xi\right)^{k}}$$

$$-\left(\beta-1\right)\nu\sum_{i=1}^{n}\frac{G\left(x_i;\xi\right)^{k}\ln\left[G\left(x_i;\xi\right)\right]}{\left[1-G\left(x_i;\xi\right)^{k}\right]\left[\ln\left(\left\{1-G\left(x_i;\xi\right)^{k}\right\}^{\nu}\right)\right]}$$

$$-\alpha\beta\nu\sum_{i=1}^{n}\frac{G\left(x_i;\xi\right)^{k}\ln\left[G\left(x_i;\xi\right)\right]}{1-G\left(x_i;\xi\right)^{k}}\left[-\ln\left(\left\{1-G\left(x_i;\xi\right)^{k}\right\}^{\nu}\right)\right]^{\beta-1}$$

$$U_{\nu}=\frac{\partial lf}{\partial \alpha}=\frac{n}{\nu}+\left(\beta-1\right)\sum_{i=1}^{n}\frac{\ln\left(1-G\left(x_i;\xi\right)^{\kappa}\right)}{\left[\ln\left(\left\{1-G\left(x_i;\xi\right)^{k}\right\}^{\nu}\right)\right]}$$

$$+\alpha\beta\sum_{i=1}^{n}\left[-\ln\left(\left\{1-G\left(x_i;\xi\right)^{k}\right\}^{\nu}\right)\right]^{\beta-1}\left[\ln\left(1-G\left(x_i;\xi\right)^{k}\right)\right]$$

and

$$U_{\xi}=\frac{\partial lf}{\partial \xi}=\sum_{i=1}^{n}\frac{g^{(\xi)}\left(x_i;\xi\right)}{g\left(x_i;\xi\right)}+\left(\kappa-1\right)\sum_{i=1}^{n}\frac{g^{(\xi)}\left(x_i;\xi\right)}{G\left(x_i;\xi\right)}$$

$$+\kappa\sum_{i=1}^{n}\frac{G\left(x_i;\xi\right)^{k-1}G^{(\xi)}\left(x_i;\xi\right)}{\left[1-G\left(x_i;\xi\right)^{k}\right]}$$

$$-\left(\beta-1\right)\nu\kappa\sum_{i=1}^{n}\frac{G\left(x_i;\xi\right)^{k-1}G^{(\xi)}\left(x_i;\xi\right)}{\left[\ln\left(\left\{1-G\left(x_i;\xi\right)^{k}\right\}^{\nu}\right)\right]\left[1-G\left(x_i;\xi\right)^{k}\right]}$$

$$-\alpha\beta\nu\kappa\sum_{i=1}^{n}\frac{\left[-\ln\left(\left\{1-G\left(x_i;\xi\right)^{k}\right\}^{\nu}\right)\right]^{\beta-1}G\left(x_i;\xi\right)^{k-1}G^{(\xi)}\left(x_i;\xi\right)}{\left[1-G\left(x_i;\xi\right)^{k}\right]}$$

Solving the equations $U_{\theta}=(U_{\alpha}, U_{\beta}, U_{\kappa}, U_{\nu}, U_{\xi T})^{T}$ simultaneously produce the maximum likelihood estimates $\hat{\theta}=\left(\hat{\alpha},\hat{\beta},\hat{\kappa},\hat{\nu},\hat{\xi}^{T}\right)^{T}$ of the parameters $q=(\alpha, \beta, \kappa, \nu, \xi_{T})^{T}$. To estimate these parameters, numerical iterative techniques can be used. The information matrix will be required for interval estimation. When some standard regularity conditions are satisfied, one can verify that $\sqrt{n}\left(\hat{\alpha},\hat{\beta},\hat{\kappa},\hat{\nu},\hat{\xi}^{T}\right)^{T}-\left(\alpha,\beta,\kappa,\nu,\xi^{T}\right)^{T}$ converges in distribution to the multivariate $N_{p+4}(0, I(\alpha, \beta, \kappa, \nu, \xi)^{-1}$ distribution, where p denotes the dimension of ξ and $I\left(\alpha,\beta,\kappa,\nu,\xi\right)=E\left[J\left(\hat{\alpha},\hat{\beta},\hat{\kappa},\hat{\nu},\hat{\xi}\right)\right]$ is the expected information matrix. The approximate multivariate normal distribution, where $J^{-1}(\theta)$ is the observed information matrix evaluated at $\left(\hat{\alpha},\hat{\beta},\hat{\kappa},\hat{\nu},\hat{\xi}\right)=\left(\alpha,\beta,\kappa,\nu,\xi\right)$ can be used to construct approximate confidence intervals for the individual parameters. An asymptotic confidence interval (ACI) with significance level for each parameter (θ) is given by

$$ACI\left(\theta_i,100\left(1-g\right)\right)\%=\left(\hat{\theta}_l-Z_{g/2}\sqrt{\hat{I}^{\theta_i,\theta_i}},\ \hat{\theta}_l-Z_{g/2}\sqrt{\hat{I}^{\theta_i,\theta_i}}\right)$$

To test, if parameter say, δ is statistically different from one, *i.e.* for testing the null hypothesis H_0: $\delta = 1$ against H_1: $\delta \neq 1$, the Likelihood Ratio (LR) statistic was used as follows. $LR=2\left\{\ell\left(\hat{\theta}\right)-\ell\left(\tilde{\theta}\right)\right\}$, where $\hat{\theta}$ is the vector of unrestricted MLEs under H_1 and $\tilde{\theta}$ is the vector of restricted MLEs under H_0. Under the null hypothesis, the limiting distribution of LR is a χ_1^2 distribution. If the test statistic exceeds the upper 100(1-α)% quantile of the χ_1^2 distribution, then the null hypothesis is rejected.

For example, consider the parent distribution as distribution with scale parameter $\lambda > 0$ and shape parameter $\theta > 0$, for testing the null hypothesis H_0: $\kappa = 1$ against H_1: $\kappa \neq 1$, we use the test statistic $LR = 2\left\{\ell\left(\hat{\alpha},\hat{\beta},\hat{\kappa},\hat{\nu},\hat{\lambda},\hat{\theta}\right)-\ell\left(\hat{\alpha},\hat{\beta},1,\hat{\nu},\hat{\lambda},\hat{\theta}\right)\right\}$. If the test statistic exceeds the upper $100(1-\alpha)$% quantile of the χ_1^2 distribution, then we reject the null hypothesis.

5. Real Life Application

In this section, the Reflected Kumaraswamy Weibull-X{Exponential} family of distribution was applied to two real life data. The purpose is to investigate the usefulness and flexibility of the new family over some other existing ones in the literature. As an example, the Reflected Kumaraswamy Weibull-Exponential Exponential was used and comparison was made with respect to the Beta-G family of distribution [Eugene *et al*. (2002)], Kumaraswamy-G family of distributions **[8]**, Exponentiated-G family of distributions [Cordeiro *et al*. (2013)] and Weibull-G family of distributions **[7]**.

Selection criteria such as the Negative Log-likelihood (NLL), Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Consistent Akaike Information Criteria (CAIC) and Hannan and Quinn

Table 1: Table of results using the diesel engine dataset.

Models	NLL	AIC	CAIC	BIC	HQIC
RKW-EE	79.7029	169.4059	171.1706	177.8503	172.4592
BeE	87.4582	180.9166	181.5832	185.9832	182.7485
KuE	83.9401	173.8804	174.5470	178.9470	175.7123
EGE	90.1426	186.2853	186.9520	191.3520	188.1173
WeE	82.4755	170.9511	171.6178	176.0177	172.7830
We	89.4965	183.1456	183.4699	186.5233	184.3669
E	113.3193	228.6385	228.7438	230.3274	229.2492

Table 2: The parameter estimates, KS, W and A statistics using the diesel engine dataset.

Models	Estimates	W	A	KS	KS-pvalue
RKW-EE	0.64257, 0.55903, 32.97255, 33.53944, 0.31487	0.02837	0.21200	0.08346	0.94329
BeE	7.67351, 18.87539, 0.05581	0.20649	1.36872	0.12918	0.51665
KuE	4.67414, 48.98003, 0.08309	0.11280	0.80474	0.11283	0.68846
EGE	2.59208, 9.51455, 0.17354	0.27574	1.76009	0.15420	0.29748
WeE	3.87231, 0.07919, 0.01144	0.07703	0.57328	0.10724	0.74700
We	0.15225, 2.20462	0.13268	0.92880	0.20237	0.07551
E	0.15993	0.20652	1.36890	0.36310	0.00005

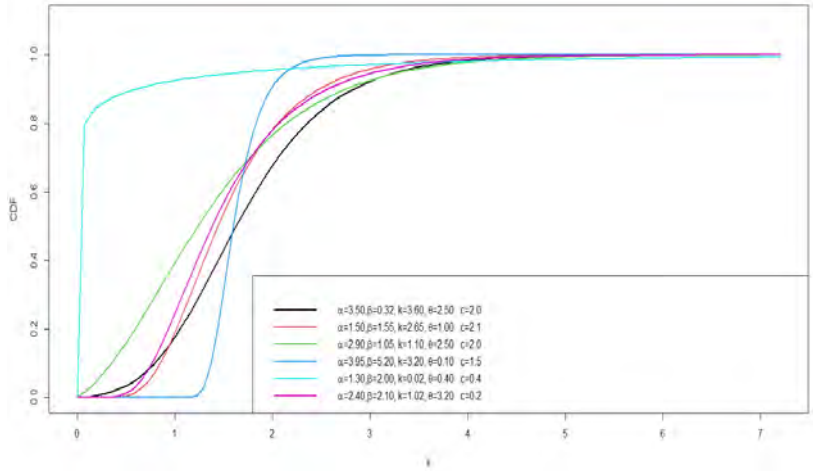

Fig. 1: The cdf plot for the Reflected Kumaraswamy Weibull-Exponential Exponential (RKW-EE) distribution

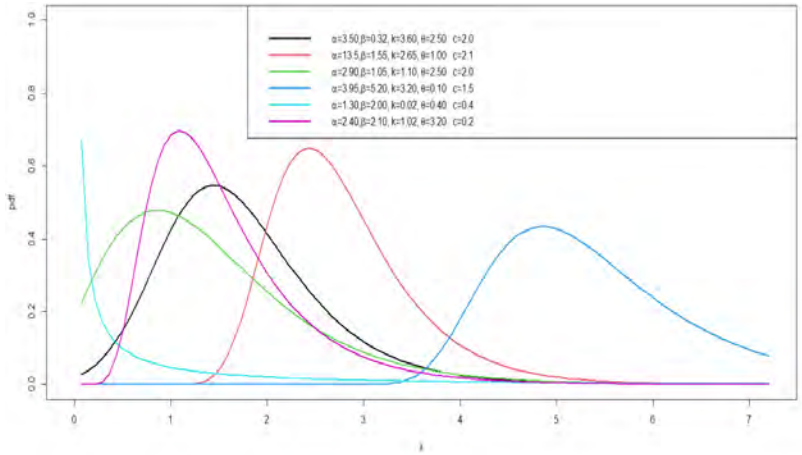

Fig. 2: The pdf plot for the Reflected Kumaraswamy Weibull-Exponential Exponential (RKW-EE) distribution

Table 3: Table of results using the glass fibre dataset.

Models	NLL	AIC	CAIC	BIC	HQIC
RKW-EE	11.4411	32.8823	33.9350	43.5980	37.0969
BeE	24.0994	54.1989	54.6057	60.6283	56.7276
KuE	17.4927	40.9854	41.3921	47.4148	43.5141
EGE	31.3835	68.7670	69.1737	75.1964	71.2957
WeE	15.2068	36.4136	36.8204	42.8430	38.9424
We	18.0609	40.1547	40.3547	44.4409	41.8405
E	88.8303	179.6606	179.7262	181.8038	180.5035

Information Criteria (HQIC) were used to select the best family or distribution. Any model that has the lowest value for these criteria was selected as the model with the best fit. Goodness of fit tests like the Kolmogorov Smirnov (KS), Shapiro- Wilk (W) and Anderson Darling (A) tests were conducted. The models compared are: Reflected Kumaraswamy Weibull-Exponential Exponential (RKW-EE) distribution, Beta Exponential (BeE) distribution, Kumaraswamy Exponential (KuE) distribution, Exponentiated Generalized Exponential (EGE) distribution, Weibull Exponential (WE) distribution, Weibull (W) distribution and Exponential (E) distribution. R software was used for the analysis with the aid of 'Newdistns' and 'AdequacyModel' packages.

First Application: The dataset on the time-to-failure for 40 suits of turbochargers in diesel engines was considered. The data has been used in Xu *et al*. (2003) and Hu *et al*. (2010). The observations are:

1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0

The dataset was analyzed, and the results are given in Table 1.

For all the five criteria shown in Table 1, the RKW-EE distribution has the lowest values. Therefore, it can be selected as the model with the best fit. The W, AD and KS statistics are provided in Table 2.

For the first application, the histogram plot is presented in Fig. 4 while the empirical cdf (ecdf) plot is

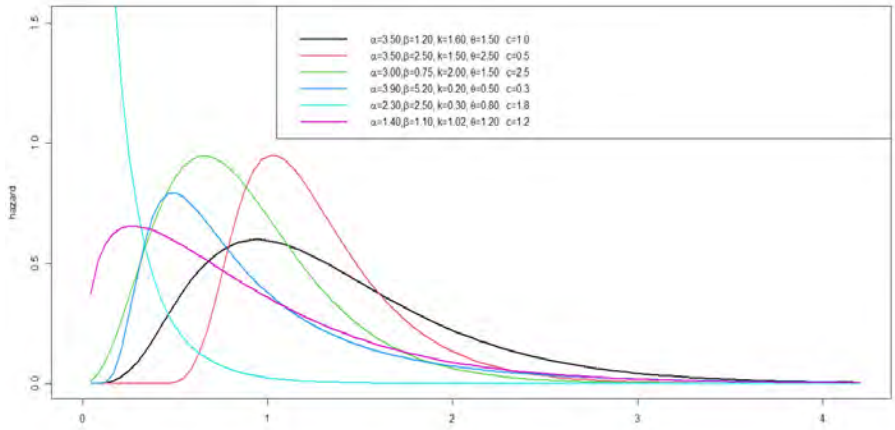

Fig. 3: The hazard plot for the Reflected Kumaraswamy Weibull-Exponential Exponential (RKW-EE) distribution

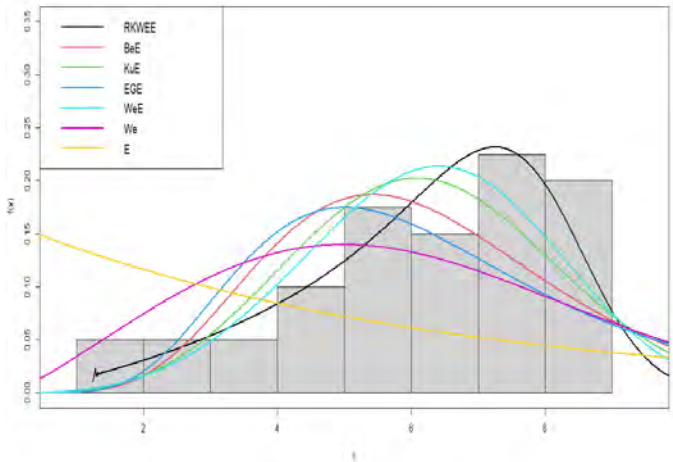

Fig. 4: Histogram plot for all the competing models using the diesel engine dataset

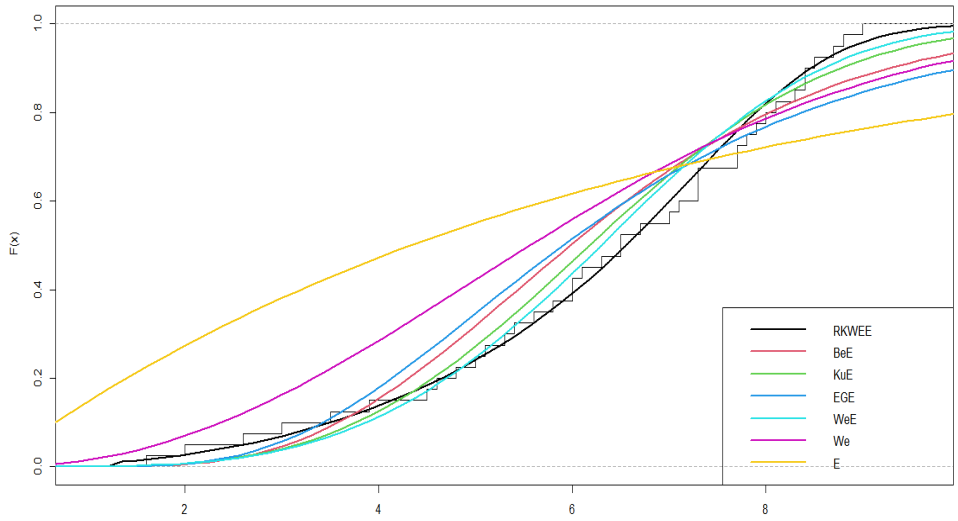

Fig. 5: The ecdf plot for all the competing models using the diesel engine dataset

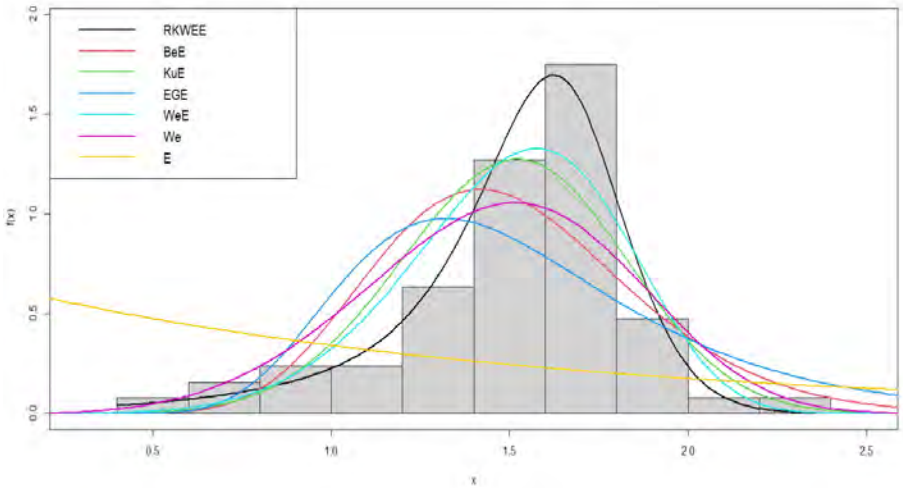

Fig. 6: The histogram plot for all the competing models using the glass fibre dataset

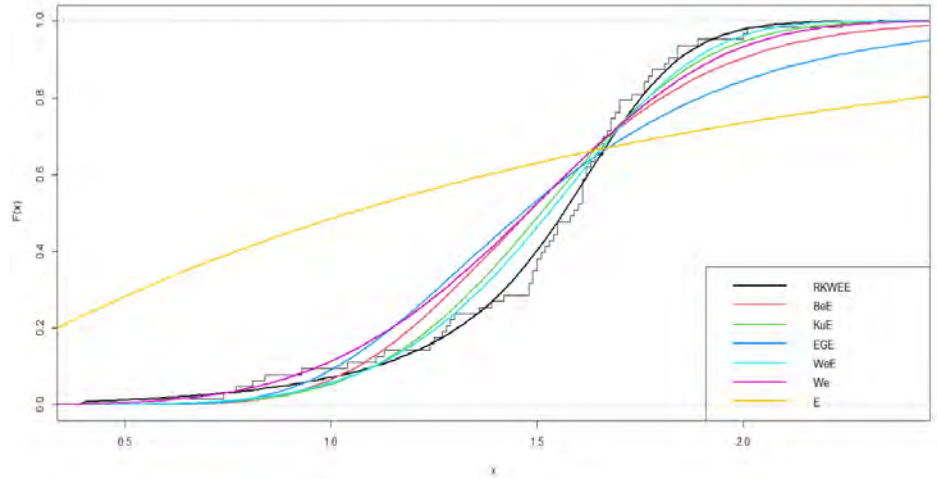

Fig. 7: The ecdf plot for all the competing models using the glass fibre dataset

presented in Fig. 5.

The plots in Fig. 5 support the ones in Fig. 4 as the cdf plot for the RKW-EE distribution fits the diesel engine dataset than the other competing models.

Second Application: This dataset relates to the strength of 1.5cm glass fibers. It has been analyzed

Table 4: The parameter estimates, KS, W and A statistics using the glass fibre dataset.

Models	Estimates	W	A	KS	KS-pvalue
RKW-EE	1.57903, 0.31729, 66.77036, 25.92920, 2.15421	0.06232	0.37767	0.08876	0.70359
BeE	17.43153, 25.39406, 0.35209	0.57314	3.14334	0.21721	0.00523
KuE	9.22466, 60.99239, 0.63255	0.33809	1.85122	0.17700	0.03859
EGE	-0.76568, 31.38352, -3.41088	0.78626	4.28754	0.22870	0.00274
WeE	5.78115, 0.32158, 0.19751	0.23722	1.30364	0.15219	0.10802
We	0.62404, 4.47677	0.29183	1.59862	0.21797	0.00502
E	0.66364	0.57020	3.12704	0.41799	5.49e-10

previously by Bourguignon *et al*. (2014), Smith and Naylor (1987), Merovci *et al*. (2016), Oguntunde *et al*. (2017). The observations are:

0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28,1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50,1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68,1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24

The dataset was analyzed and the results are given in Table 3.

The results in Table 3 is in agreement with the ones in Table 1 as the RKW-EE distribution performs better than the other competing models. The result for the corresponding goodness of fit test is provided in Table 4.

The histogram plot is presented in Fig, 6, while the empirical cdf (ecdf) plot is presented in Fig. 7.

In Fig. 6, the curve which corresponds to the RKW-EE distribution has the highest peak and fits the histogram of the dataset better than the other competing models. The corresponding ecdf plot is presented in Fig. 7.

The plots in Fig. 7 support the ones in Fig. 6 as the cdf plot for the RKW-EE distribution fits the glass fibre dataset than the other competing models.

6. Conclusion

The Reflected Kumaraswamy Weibull-X{Exponential} (RKW-X{Exponential}) family of distribution has been developed in this research. Various general statistical properties of the family have been established and special cases have been identified. The model parameters were estimated using the maximum likelihood method of estimation and applications to real life datasets were provided. The RKW-X{Exponential} family of distribution was compared with other existing families like the Beta-G, Kumaraswamy-G, Exponentiated-G and Weibull-G families of distributions. For the applications provided, the RKW-X{Exponential} family shows a greater potential than the other families of distributions. The RKW-X is robust and flexible and it can serve as an alternative method for developing compound probability models with good modeling capabilities. Future research would include exploring simulation studies for the RKW-X{Exponential} family of distributions using one of its special models as a case study.

References

Alizadeh, M., E. Altun, A.Z. Afify and G. Ozel (2019). The extended odd Weibull-G family: Properties and applications. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, **68(1)**, 161-186.

Alizadeh, M., G.M. Cordeiro, L.G.B. Pinho and I. Ghosh (2017). The Gompertz-G family of distributions. *J. Stat. Theory and Practice*, **11(1)**, 179-207.

Alizadeh, M., M.H. Tahir, G.M. Cordeiro, M. Mansoor, M. Zubair and G.G. Hamedani (2015). The Kumaraswamy Marshall-Olkin family of distributions. *J. Egypt. Math. Soc.*, **23(3)**, 546-557.

Al-Shomrani, A., O. Arif, A. Shawky, S. Hanif and M.Q. Shahbaz (2016). Topp-Leone family of distributions: Some properties and applications. *Pak. J. Statistics and Operation Res.*, **12(3)**, 443-451.

Alzaatreh, A., C. Lee and F. Famoye (2013). A new method for generating families of continuous distributions. *Metron*, **71(1)**, 63-79.

Aslam, M., Z. Hussain and Z. Asghar (2018). Cubic transmuted-G family of distributions and its properties. *Stochastic and Quality Control*, **33(2)**, 103-112.

Bourguignon, B.M., R. Silva and G.M. Cordeiro (2014). The Weibull-G family of probability distributions. *J. Data Sci.*, **12**, 53-68.

Cordeiro, G.M. and M. de Castro (2011). A new family of

generalized distributions. *J. Stat. Computation and Simulation*, **81(7)**, 883-898.

Cordeiro, G.M., E.M.M. Ortega and D.C.C. da Cunha (2013). The Exponentiated Generalized Class of Distributions. *J. Data Sci.*, **11**, 1-27.

Dinesh, K., A.S. Yadav, P. Kumar, P. Kumar, S.K. Singh and U. Singh (2021). Transmuted Inverse Lomax Distribution and its Properties. *Int. J. Agricult. Stat. Sci.*, **17(1)**, 1-8.

Eugene, N., C. Lee and F. Famoye (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and Methods*, **31**, 497-512.

Groeneveld, R.A. and G. Meeden (1984). Measuring Skewness and Kurtosis. *The Statistician*, **33**, 391-399.

Gupta, R.D. and D. Kundu (2001). Exponentiated exponential family: an alternative to gamma and Weibull. *Biometrical Journal*, **43(1)**, 117-130.

Hu, C., X. Si and J. Yang (2010). Dynamic evidential reasoning algorithm for systems reliability prediction. *Int. J. Systems Sci.*, **41(7)**, 783-796.

Keller, A.Z. and A.R. Kamath (1982). Alternative reliability models for mechanical systems. *Third International Conference on Reliability and Maintainability*. Toulouse, France.

Kenney, J.F. and E.S. Keeping (1962). *Mathematics of Statistics*. Part 1, 3rd Edition, Princeton, New Jersey.

Khaleel, M.A., P.E. Oguntunde, J.N. Al Abbasi, N.A. Ibrahim and M.H. AbuJarad (2020). The Marshall-Olkin Topp Leone-G family of distributions: A family for generalizing probability models. *Scientific African*, **8**, e00470.

Madhulika, D., S. Vishwajee and S.K. Yadav (2020). Assessment of the regional disparities in development of Agricultural sector in Uttar Pradesh: A statistical analysis. *Int. J. Agricult. Stat. Sci.*, **16(2)**, 617-624.

Merovci, F., M.A. Khaleel, N.A. Ibrahim and M. Shitan (2016). The beta type X distribution: properties with application. *SpringerPlus*, **5(1)**, 697. DOI: 10.1186/s40064-016-2271-9

Moors, J.J.A. (1988). A quantile alternative for kurtosis. *J. Royal Statistical Society-D*, **37**, 25-32.

Mudholkar, G.S., D.K. Srivastava and M. Freimer (1995). The exponentiated Weibull family. *Technometrics*, **37**, 436-445.

Nadarajah, S. and S. Kotz (2006). The exponentiated type distributions. *Acta Applicandae Mathematicae*, **92**, 97-111.

Oguntunde, P.E., M.A. Khaleel, M.T. Ahmed, A.O. Adejumo and O.A. Odetunmibi (2017). A new generalization of the Lomax distribution with increasing, decreasing, and constant failure rate. *Modelling and Simulation in Engineering*, https://doi.org/10.1155/2017/6043169

Silva, R., F. Gomes-Silva, M. Ramos, G. Cordeiro, P. Marinho and T.A.N. de Andrade (2019). The Exponentiated Kumaraswamy-G Class: General Properties and Application. *Revista Colombiana de Estadistica*, **42(1)**, 1-33.

Smith, R.L. and J.C. Naylor (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. *Applied Statistics*, **36**, 258-369.

Wallodi, W. (1951). A Statistical Distribution Function of Wide Applicability. *J. Applied Mechanics*, **18**, 293-297.

Xu, K., M. Xie, L.C. Tang and S. L. Ho (2003). Application of Neural Networks in Forecasting Engine Reliability. *Applied Soft Computing*, **2**, 255-268.