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ORIGINAL ARTICLE

A NEW METHOD FOR CONSTRUCTION OF UN-EQUALLY REPLICATED PAIRWISE BALANCED DESIGN

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Abstract : A new method has been proposed for the construction of un-equally replicated Pairwise Balanced Design with repeated blocks. The method discusses the construction of Pairwise Balanced Designs with repeated blocks by reinforcing some treatments in BIBDs of the series $v = s$, $b = {}^{v}C_2$, $r = v - 1$, $k = 2$, $\lambda = 1$ add r more blocks. In each of these r-blocks, the first (v-1) treatments appear 1 time and the *v*th treatment appears r-times. Where, s is any positive integer (≥ 4). The methods were illustrated with suitable numerical examples. Optimality criteria's have been derived and the constructed design was found to be universally optimum.

***Key words* :** Balanced design, Connectedness, Variance Balanced Design, Pairwise Balanced Design, Mutually Orthogonal Design

Mathematics Subject Classification: Computing Classification System

1. Introduction

In a Pairwise balanced block design, any pair of treatments within the blocks occur equally often. The literature on combinatorial theory contains many contributions to the existence and construction of Pairwise balanced designs, including for settings with unequal block sizes. For example, Bose and Shrikhande (1960) used the pairwise balanced design in the context of constructing Mutually Orthogonal Latin Square (MOLS). The literature on combinatorial theory contains many contributions to the existence and construction of Pairwise balanced designs with un-equal block sizes.

Hedayat and Stufken (1989) showed that the problems of constructing Pairwise balanced designs and Variance balanced block designs are equivalent.

Effanga *et al*. (2009) developed a non-linear non-pre-emptive binary integer goal programming model for the construction of D-optimal pairwise balanced incomplete block designs. It was observed that the design was not unique under the same set of parameters and that alternative designs could be obtained from a given design by interchanging rows or columns of the corresponding design matrix.

Bose and Shrikhande (1960) defined Pairwise balanced design as following:

Definition 1.1: An arrangement of v treatments in b blocks is defined as Pairwise balanced design of index λ of type (v; $k_1, k_2, ..., k_m$) provided.

a) Each set contains ($k_1, k_2, ..., k_m$) symbols that are all distinct.

b) $k_i \leq v$; $k_i \neq k_j$ and

c) every pair of distinct treatments occurs in exactly λ sets of the design.

Further, they also showed some of the parametric relation of Pairwise balanced design, which is namely

$$b = \sum_{i=1}^{m} b_i \text{ and } \lambda(v-1) = \sum_{i=1}^{m} b_i k_i (k_i - 1)$$

A characterization of Pairwise balanced design in terms of NN^T matrix can be expressed in the following way:

A block design D is called Pairwise balanced design, if all the off diagonal elements of the concurrent matrix NN^T are same (constant).

$(r-\lambda)\, Iv + \lambda\, E_{vv}$

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where, I_v is a identity matrix of order $(v \times v)$ and $E_{v \times v}$ is the unit matrix of order $v \times v$.

The above equation can be rewritten as

$$NN^T = Diag(r - \lambda) + \lambda + E_{vv}$$

Ghosh and Desai (2014) developed some new equally replicated Pairwise balanced design using triangular PBIBD design and also obtained optimality. Rajarathinam *et al*. (2016), gave the procedure for construction of equally replicated Pairwise balanced design using factorial design. So far in the literature, different methods for construction of equally replicated Pairwise balanced design have been discussed, but in this paper a new method has been proposed for the construction of un-equally replicated Pairwise balanced design with repeated blocks.

Das (2002) carried out optimality criteria and some results on optimal block designs. In this paper, we have used optimal criteria of A, D and E optimal block design. Moreover, it is shown that pairwise balanced design constructed here are A, D and E optimality and hence, it is also Universal optimal.

2. Methods of Construction

Theorem 2.1: Consider a BIBD design D with the parameters v, $b = {}^vC_2$, $r = v - 1$, $k = 2$, $\lambda = 1$. Add r more blocks. In each of these r-blocks, the first v-1 treatments appear 1 time and the vth treatment appears r-times. Then, we get the Pairwise balanced design with repeated blocks having the parameters $v^* = v$, $r^* = (2r, vr)$, $b^* = b + r$, $k^* = (k, v + r - 1)$, $\lambda_1 = v$, $\lambda_2 = v^2 - 2v + 2$.

Now consider the incidence matrix N defined as

$$N^* = \begin{bmatrix} N_{v \times b} \vdots E_{v-1xv-1} \\ \vdots rJ^{\prime}_{v-1} \end{bmatrix}_{v \times (b+r)} \tag{1}$$

Proof : The incidence matrix given in (1) consists of v rows and considering these as treatments obviously the number of treatments are v. Also it is obvious that, the above incidence matrix consists of $b = {}^vC_2$ columns and considering these as the blocks the numbers of blocks are $b = {}^vC_2$.

The concurrent matrix NN^T is computed as

$$NN^T = \begin{bmatrix} E_{(v-1)\times(v-1)} & F_{(v-1)\times 1} \\ F^T_{1\times(v-1)} & G_{(1\times 1)} \end{bmatrix}$$

where,

$$E_{(v-1)\times(v-1)} = \begin{bmatrix} r_1 & \lambda_1 & \cdots & \lambda_1 \\ \lambda_1 & r_1 & \cdots & \lambda_1 \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1 & \lambda_1 & \cdots & r_1 \end{bmatrix}$$

$$F_{(v\times 1)} = \begin{bmatrix} \lambda_2 \\ \vdots \\ \lambda_2 \end{bmatrix}; F^T_{(1\times v)} = [\lambda_2, \ldots, \lambda_2]; G_{(b\times 1)} = \lambda_2(v-1)$$

Also, $NN^T = (r_1 - \lambda_1)I_v + \lambda_1 E_{vv} + T$

where,

$$T = \begin{bmatrix} r_2 - (v-1)\lambda_1 & r_2 - (v-1)\lambda_1 & \cdots & (\lambda_2 - v) \\ r_2 - (v-1)\lambda_1 & r_2 - (v-1)\lambda_1 & \cdots & (\lambda_2 - v) \\ \vdots & \vdots & \cdots & \vdots \\ r_2 - (v-1)\lambda_1 & r_2 - (v-1)\lambda_1 & \cdots & (v-2)r_2 \end{bmatrix}_{(v\times v)}$$

Hence, this design is the un-equally replicated Pairwise balanced design with the following parameters $v^* = v$; $b^* = b + r$;

$$\underline{r} = \left\{(2s-2)\underline{1}^{\prime}_{(s-1)}, (s-1)\right\} \text{ (or) } r^* = (2r, vr)$$

$$\underline{k} = \left\{2\underline{1}^{\prime}_{\frac{s(s+1)}{2}}, 2(s-1)\underline{1}^{\prime}_{(s-1)}\right\} \text{ (or) } k^* = (k, v+r-1)$$

2.1 Numerical illustration

For $s = 4$, consider BIB design, D of the series $v^* = v$; $b^* = b + r$;

$$\underline{r} = \left\{(2s-2)\underline{1}^{\prime}_{(s-1)}, s(s-1)\right\}; \underline{k} = \left\{2\underline{1}^{\prime}_{\frac{s(s+1)}{2}}, 2(s-1)\underline{1}^{\prime}_{(s-1)}\right\}.$$

Then the following incidence matrix N gives the un-equally replicated Pairwise balanced block design with the parameters, $v = 4$, $r_1 = 6\underline{1}^{\prime}_3$; $r_2 = 12$; $\lambda_1 = 4$; $\lambda_2 = 10$.

$$N = \left[\begin{array}{cccccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 1 & 3 & 3 & 3 \end{array}\right]_{(4\times 9)} \tag{2}$$

The NN^T matrix is computed as

$$NN^T = \left[\begin{array}{ccc|c} 6 & 4 & 4 & 10 \\ 4 & 6 & 4 & 10 \\ 4 & 4 & 6 & 10 \\ \hline 10 & 10 & 10 & 30 \end{array}\right]_{(4\times 4)}$$

This can be written as

$$NN^T = 21_{(4\times 4)} + 4E_{(4\times 4)} + \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 6 \\ 6 & 6 & 6 & 24 \end{bmatrix}_{(4\times 4)}$$

Hence, the incidence matrix (2) gives un-equally replicated pairwise balanced design with parameters v = 4, b = 9, $r_1 = 6\underline{1}'_3$; r_2 = 12; λ_1 = 4; λ_2 = 10. A list of non-zero eigen values of pairwise balanced design along with its parameters is given in Table 1.

2.2 Optimality Criteria

The number of replications and block size of the design D of the N matrix is given by

$$\underline{r} = \left\{(2s-2)\underline{1}'_{(s-1)}, s(s-1)\right\}$$

$$\underline{k} = \left\{2\underline{1}'_{\frac{s(s+1)}{2}}, 2(s-1)\underline{1}'_{(s-1)}\right\}$$

The computed $NK^{-1}N^T$ matrix is

$$\left[\begin{array}{cccccc|c} \alpha_1 & \beta_1 & \beta_1 & \cdots & \beta_1 & & \gamma_1 \\ \beta_1 & \alpha_1 & \beta_1 & \cdots & \beta_1 & & \gamma_1 \\ \beta_1 & \beta_1 & \alpha_1 & \cdots & \beta_1 & & \gamma_1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & & \vdots \\ \beta_1 & \beta_1 & \beta_1 & \cdots & \alpha_1 & & \gamma_1 \\ \hline \gamma_1 & \gamma_1 & \gamma_1 & \cdots & \gamma_1 & & \delta_1 \end{array}\right]_{(s\times s)}$$

where, $\alpha_1 = \frac{s}{2}; \beta_1 = 1; \gamma_1 = \frac{s}{2}; \delta_1 = \frac{s(s-1)}{2}$

The computed $C = R - NK^{-1}N^T$ matrix is

$$\begin{bmatrix} \alpha_2 & \beta_2 & \beta_2 & \cdots & \beta_2 & \gamma_2 \\ \beta_2 & \alpha_1 & \beta_2 & \cdots & \beta_2 & \gamma_2 \\ \beta_1 & \beta_2 & \alpha_1 & \cdots & \beta_2 & \gamma_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \beta_2 & \beta_2 & \beta_2 & \cdots & \alpha_2 & \gamma_2 \\ \gamma_2 & \gamma_2 & \gamma_2 & \cdots & \gamma_2 & \delta_2 \end{bmatrix}_{(s\times s)}$$

where, $\alpha_2 = \frac{3s-4}{2}; \beta_2 = -\beta_1; \gamma_1 = -\gamma_2$

The non-zero eigen values of the above C-matrix is

$$\theta_1 = \left(\frac{3s-2}{s-(s-2)}\right) \text{ with multiplicity (v-2)}$$

$$\theta_2 = \left(\frac{s^2}{s-(s-2)}\right) \text{ with multiplicity 1.}$$

Trace of the C matrix = $2(s-1)^2$.

A-optimality

Let $\theta_1, \theta_2, \theta_3, \ldots, \theta_{(v-1)}$ be non-zero eigen values with multiplicities (v-1) of C_d matrix of design d. Pairwise balanced design with unequal block size will be A- optimal, if

$$\sum_{i=1}^{(v-1)} \frac{1}{\theta_i} \geq \frac{(v-1)^2}{tr(C_d)} \tag{3}$$

where, $tr(C_d) = 2(s-1)^2$.

Solving (3), we obtain that pairwise block design will be A-optimal if

$$\sum_{i=1}^{(v-1)} \frac{1}{\theta_i} \geq \frac{1}{2} \text{ hold true.}$$

D-optimality

Let $\theta_1, \theta_2, \theta_3, \ldots, \theta_{(v-1)}$ be non-zero eigen values with multiplicities (v-1) of C_d matrix of design d.

Table 1: Parameters of Pairwise Balanced Designs which are A, D and E optimal.

s	Parameters of Pairwise Balanced Design								Eigen Value	
	v	b	r		k		λ		θ_1	θ_2
			r_1	r_2	k_1	r_2	λ_1	λ_2		
4	4	9	6	12	2	6	4	10	5.00(2)	8.75(1)
5	5	14	8	20	2	8	5	17	6.50(3)	12.50(1)
6	6	20	10	30	2	10	6	26	8.00(4)	18.00(1)
7	7	27	12	42	2	12	7	37	9.50(5)	24.50(1)
8	8	35	14	56	2	14	8	50	11.00(6)	32.00(1)
9	9	44	16	72	2	16	9	65	12.50(7)	40.50(1)
10	10	54	18	90	2	18	10	82	14.00(8)	50.00(1)

Pairwise balanced design with unequal block size will be *D-optimal*, if

$$\left(\prod_{i=1}^{(v-1)} \frac{1}{\theta_i}\right) \leq \left(\frac{tr(C_d)}{(v-1)}\right)^{(v-1)} \tag{4}$$

Solving (4), we obtain that pairwise block design will be D-optimal, if

$$\left(\prod_{i=1}^{(v-1)} \frac{1}{\theta_i}\right) \leq \left(\frac{2(s-1)^2}{(s-1)}\right)^{s-1} \text{ hold true.}$$

E-optimality

Let θ_1, θ_2, θ_3, ..., $\theta_{(v-1)}$ be non-zero eigen values with multiplicities (v-1) of C_d matrix of design d. Pairwise balanced design with unequal block size will be *E-optimal*, if

$$Min(\theta_i) \leq \frac{tr(C_d)}{v-1} \tag{5}$$

where, $tr(C_d) = 2(s-1)^2$

Solving (5), we obtain that pairwise block design will be *E-optimal* if,

$$Min(\theta_i) \leq \frac{2(s-1)^2}{s-1} \text{ holds true.}$$

Since pairwise balanced designs are A, D and E optimal. Hence, pairwise balanced design is universal optimal.

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